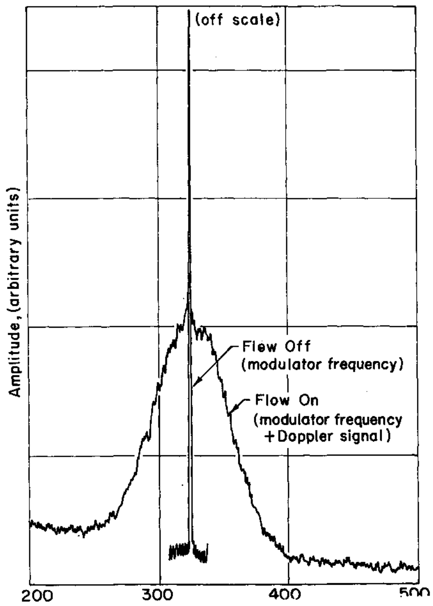

Fig. 3 Typical radial turbulence signal, $v_M = 325$ kHz.

simultaneous measurement of u' and v', multiplication, and averaging, is not necessary.

Apparatus and results: The newly developed velocimeter can measure mean flow components and turbulence quantities $\langle u'^2 \rangle$, $\langle v'^2 \rangle$, and $\langle u'v' \rangle$ with minimal adjustment of only one optical component, a radial diffraction grating, serving as a combination beam splitter, measurement direction selector, and frequency modulator. The apparatus is diagramed in Fig. 1. Two of the diffraction orders are selected by a mask and focused in the test section with a single converging lens guaranteeing "self alignment," with beams focusing at the same point. Although there are many possibilities, it has been convenient to use a "reference mode" system with the zero-order (undeflected) beam serving as the reference beam so that alignment through the fixed lens, apertures, and photomultiplier is independent of translation of the grating.

Different directions of measurement (various ϕ) are selected by translating the diffraction grating parallel to itself, and locating the incident laser beam at different positions around the circumference of the wheel at constant radius.

Most measurements are made with the grating stationary, although by rotating the grating any turbulent spectrum can

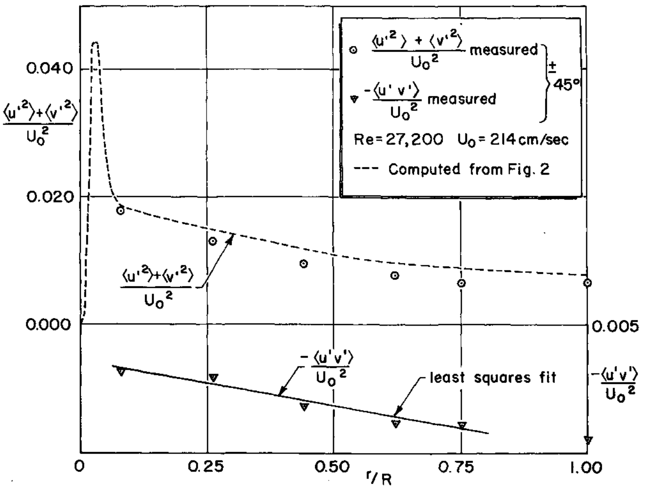

Fig. 4 Reynolds stress and turbulence energy across pipe.

be shifted in frequency by 84.2 Hz/rpm. This biasing is crucial to obtain $\langle v'^2 \rangle$ ($\phi = 90°$) and also useful to bring other Doppler signals into more convenient frequency ranges.

Representative turbulence measurements made in a $\frac{1}{2}$ inch-square lucite pipe ($l/d = 65$) are presented. Axial and radial fluctuations across the pipe are shown in Fig. 2. Turbulent broadening about the biasing frequency is shown in Fig. 3 in a typical determination of $\langle v'^2 \rangle$. Figure 4 presents measurements of $\langle u'v' \rangle$ across the pipe and the corresponding turbulent energy from Eq. (4). Secondary (radial) flow was also measured, at maximum about 0.5% of the centerline axial velocity.

References

[1] Goldstein, R. J. and Hagan, W. F., "Turbulent Flow Measurements Utilizing the Doppler Shift of Scattered Laser Radiation," *The Physics of Fluids*, Vol. 10, No. 6, June 1967, pp. 1349–1352.

[2] Penner, S. S., "Use of Lasers for Local Measurements of Velocity Components, Species Densities and Temperatures," AIAA Paper 71-283, Albuquerque, N. Mex., 1971.

[3] Goldstein, R. J. and Kreid, D. K., "Measurement of Laminar Flow Development in a Square Duct Using a Laser-Doppler Flowmeter," *Journal of Applied Mechanics*, Ser. E, Vol. 34, No. 4, Dec. 1967, pp. 813–818.

[4] Brayton, D. B., "A Simple Laser, Doppler Shift, Velocity Meter with Self Aligning Optics," Presented at the Electro-Optical Systems Design Conference, Sept. 16–18, 1969, New York.

[5] Rudd, M. J., "A New Theoretical Model for the Laser Dopplermeter," *Journal of Scientific Instruments* (*Journal of Physics E*) Ser. 2, Vol. 2, 1969, pp. 55–59.

[6] Cummins, H. Z., Knable, N., Gampel, L., and Yeh, Y., "Frequency Shifts in Light Diffracted by Ultrasonic Waves in Liquid Media," *Applied Physics Letters*, Vol. 2, No. 3, Feb. 1963, pp. 62–64.

[7] Stevenson, W. H., "Optical Frequency Shifting by Means of a Rotating Diffraction Grating," *Applied Optics*, Vol. 9, No. 3, March 1970, pp. 649–652.

[8] Denison, E. B. and Stevenson, W. H., "Oscillatory Flow Measurements with a Directionally Sensitive Laser Velocimeter," *Review of Scientific Instruments*, Vol. 41, No. 10, Oct. 1970, pp. 1475–1478.

Buckling of a Circular Cylindrical Shell in Axial Compression and SS4 Boundary Conditions

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The influence of boundary conditions on the buckling loads of circular cylindrical shells is well known and has received much attention in literature. Considering the buckling of a finite cylinder in axial compression, one finds that for most boundary conditions only numerical data is available. Closed form solutions exist only for the classical SS3 boundary condition, e.g.,[1] and for the SS1 boundary condition.[2] For a semi-infinite cylinder there exists a closed form solution also for the SS2 boundary condition.[3] The purpose of this Note is to give closed form bounds to the buckling problem of a circular cylindrical shell in axial compression for the SS4 boundary conditions. Although the solution is approximate, it is shown that the true solution is bounded between two very close quantities.

The Donnell equations,[4] for axial compression, are

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$$u_{,xx}+\frac{1-v}{2}u_{,\phi\phi}+\frac{1+v}{2}v_{,x\phi}-vw_{,x}=0 \quad (1a)$$

$$\frac{1+v}{2}u_{,x\phi}+\frac{1-v}{2}v_{,xx}+v_{,\phi\phi}-w_{,\phi}=0 \quad (1b)$$

$$\nabla^4 w+12(R/h)^2(w-vu_{,x}-v_{,\phi})+4K^2\rho w_{,xx}=0 \quad (1c)$$

Where u, v, w are the nondimensional displacements, x, ϕ, the axial and circumferential nondimensional coordinates, L is the length of the shell, R its radius and h its thickness. v is Poisson's ratio, $K^4=3(1-v^2)(R/h)^2$, and ρ is the ratio (eigenvalue) between the buckling stress and the classical buckling stress.

The SS4 boundary condition is

$$w=M_x=u=v=0 \quad \text{at} \quad x=0, \quad L/R \quad (2)$$

Noting the similarity between the SS3 and the SS4 boundary conditions, the displacement field is taken in the form:

$$u=[U(x)+A\cos k\beta x]\cos n\phi \quad (3a)$$

$$v=[V(x)+B\sin k\beta x]\sin n\phi \quad (3b)$$

$$w=C\sin k\beta x\cos n\phi$$

where $\beta=\pi R/L$, k is the number of half waves in the axial direction, A, B, C, are constants and $U(x)$, $V(x)$, are to be regarded as correction functions.

Now, one can express A, B, in terms of C, and determine $U(x)$, $V(x)$, in such a way that Eqs. (1a) and (1b) together with the boundary conditions (2) will be satisfied. Details may be found in Refs. 5 and 6.

After the displacements (3a–c) have been determined up to the constant C, the Galerkin method (which is, in this case, equivalent to the Rayleigh-Ritz procedure) is applied to Eq. (1c)

$$\int_0^{2\pi}\int_0^{L/R}\left[\nabla^4 w+12\left(\frac{R}{h}\right)^2(w-vu_{,x}-v_{,\phi})+4K^2\rho w_{,xx}\right]w\,dx\,d\phi=0 \quad (4)$$

Performing the integration, yields

$$2\rho=Z+1/Z+\psi\theta \quad (5)$$

where

$$Z=[n^2+(k\beta)^2]^2/2K^2(k\beta)^2 \quad (6)$$

$$\theta=\frac{K^2}{\lambda^2}\frac{32}{(1+v)^2}\cdot\frac{[n^2-v(k\beta)^2]^2}{(n^2+(k\beta)^2]^4} \quad (7)$$

$$\psi=\begin{cases}\dfrac{t_n\lambda\cosh^2 t_n\lambda}{\dfrac{3-v}{1+v}\sinh 2t_n\lambda-2t_n\lambda} & \text{symmetric buckling, } k \text{ odd}\\[2ex] \dfrac{t_n\lambda\sinh^2 t_n\lambda}{\dfrac{3-v}{1+v}\sinh 2t_n\lambda+2t_n\lambda} & \text{antisymmetric buckling, } k \text{ even}\end{cases} \quad (8)$$

and

$$t_n^2=n^2/2K^2, \quad \lambda=[1/(2)^{1/2}][3(1-v^2)]^{1/4}L/(Rh)^{1/2} \quad (9)$$

In Eq. (5) the magnitude $Z+1/Z$ is recognized to be the expression for 2ρ at the SS3 boundary condition. The additional term $\psi\theta$ will be shown to be negligible. It is observed that Eq. (5) is an upper bound for the true value which is denoted by $2\rho^*$. A lower bound for $2\rho^*$ is that of the SS3 boundary condition as explained in Ref. 3. Hence

$$Z+1/Z<2\rho^*<Z+1/Z+\psi\theta \quad (10)$$

Since $t_n\lambda=(n/2)L/R$ one may replace, for usual shells, ψ by $[(1+v)/(3-v)]t_n\lambda/2$ in both the symmetrical and antisymmetrical buckling. Regarding Z as a continuous parameter, $Z+1/Z$ takes its minimum value for $Z=1$ which corresponds to $(k\beta)^2\approx 2K^2\gg n^2$. Substituting this value into Eq. (7), the result is

$$\psi\theta=\frac{4}{3(3-v)(1+v)(1-v^2)}\frac{nh^2}{RL}\frac{(v-t_n^2)^2}{(1+t_n^2)^4} \quad (11)$$

Hence, from Eq. (10),

$$2<2\rho^*<2+0(h^2/RL) \quad (12)$$

Thus, for all practical purposes, $\rho^*=1$ can be taken for the SS4 boundary condition. This is confirmed numerically by Ref. 7. Finally, it is noted that a result similar to Eq. (12) is obtained also in the axially symmetric buckling where $n=0$.

References

[1] Timoshenko, S. P. and Gere, J. H., *Theory of Elastic Stability*, 2nd ed., McGraw-Hill, New York, 1961.

[2] Hoff, N. J., "Low Buckling Stresses of Axially Compressed Circular Cylindrical Shells of Finite Length," *Journal of Applied Mechanics*, 1965, pp. 533–541.

[3] Hoff, N. J. and Rehfield, L. W., "Buckling of Axially Compressed Circular Cylindrical Shells at Stresses Smaller than the Classical Critical Value," *Journal of Applied Mechanics*, 1965, pp. 542–546.

[4] Donnell, L. H., "Stability of Thin Walled Tubes under Torion," Rept. 479, 1933, NACA.

[5] Baruch, M. and Harari, O., "Buckling of Cylindrical Shells with Axial Surface Tractions," *Journal of Applied Mechanics*, 1969, pp. 350–352.

[6] Durban, D., "On the Buckling of a Circular Cylindrical Shell under Non-uniform Load and Various Boundary Conditions," M.Sc. thesis, 1971 (in Hewbrew), Technion, Israel Instute of Technology, Haifa, Israel; also TAE Rept. 142, to be published.

[7] Hoff, N. J. and Soong, T. C., "Buckling of Circular Cylindrical Shells in Axial Compression," *Int. S. Mech. Sci.*, 7, 1964, pp. 489–520.

Calculations of the Turbulent Boundary Layer in Supersonic Nozzles

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Nomenclature

a = speed of sound
A^* = throat area
$A(x)$ = area of a cross section a distance x from the throat
C_f = local skin-friction coefficient
g_w = normalized wall enthalpy h_w/h_o
H = form factor $\delta^*(x)/\theta$
H_i = incompressible flow form factor
j = 0, 1 for two-dimensional and axisymmetric flow, respectively
L = length of the nozzle
M_e = Mach number at the edge of the boundary layer
$M_e(x)$ = average Mach number in the potential core at a section a distance x from the throat
M_{eL} = measured nozzle exit Mach number
$M_b(x)$ = one-dimensional flow Mach number based on the physical boundary at a distance x from the throat
M_{bL} = nozzle exit value of $M_b(x)$
R = radius of the axisymmetric body
T = temperature
$\bar{T}$ = reference temperature
x = coordinate along the body surface
γ = ratio of specific heats (taken as 1.4 for air)
$\delta^*(x)$ = boundary-layer displacement thickness at a section a distance x from the throat
θ = boundary-layer momentum thickness
Θ = transformed momentum thickness $\theta(T_e/T_o)^{(\gamma+1)/(2(\gamma-1)}$
μ = coefficient of viscosity
$\bar{\mu}$ = viscosity evaluated at the reference temperature

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